Lecture 25 (3/9/22)

Recall. If f has a pole at $z=a$, then it has a Laurent Series Expansion

$$f(z)=\underbrace{\sum_{j=1}^{m} c_j \frac{1}{(z-a)^j}}_{\substack{\text{singular part } S(z),\\ \text{rational w/ pole at } a.}} + \underbrace{\sum_{j=0}^{\infty} b_j (z-a)^j}_{\substack{\text{analytic in}\\ \text{a nbhd of } a.}}$$

Thus, if $f \in \mathcal{M}(G)$ and has finite # of poles $a_1,\dots,a_n$, then $\exists$ rational functions $S_1(z),\dots,S_n(z)$

$$S_k(z)=\sum_{j=1}^{m_k} c_{kj}\frac{1}{(z-a_j)^k}$$

s.t. $f-\sum_{k=1}^{n} S_k$ is analytic in G.

As in prescribing zeros, the question about ∞ number of poles arises:

① Suppose $\{a_k\}_{k=1}^{\infty}$ is a seq. of pts in G, without limit pts in G. Does there exist a meromorphic fcn w/ poles (of order according to multiplicity of the points in $\{a_k\}_{k=1}^{\infty}$)?

<u>Answer:</u> Use Weierstrass to construct $g \in H(G)$ w/ zeros at $\{a_n\}$. Then, $f = \frac{1}{g}$ has poles of prescribed order. Indeed, if g has zero at $z=a$ of order m, then

$$g(z) = \sum_{j=m}^{\infty} b_j (z-a)^j = \underset{\neq 0}{\underset{\uparrow}{b_m}}(z-a)^m + \dots$$

$$\Rightarrow \quad f(z) = \frac{1}{b_m (z-a)^m}\left(1 + c_1 (z-a) + \dots\right)$$

$$= \underbrace{\frac{1}{b_m}\frac{1}{(z-a)^m} + \frac{c_1}{b_m}\frac{1}{(z-a)^{m-1}} + \dots}_{S(z)} + \sum_{j=0}^{\infty} b_j (z-a)^j$$

Thus, Weierstrass allows us to prescribe poles and orders, but what about full singular parts?

(2) Suppose $\{a_k\}_{k=1}^{\infty}$ is a seq. of points in G and for each k, a singular part

$$S_k(z) = \sum_{j=1}^{m_k} b_j \frac{1}{(z-a_k)^j}, \quad k=1,\dots$$

is given. Does there exist $f \in \mathcal{M}(G)$ w/ poles and singular parts given by $\{a_k\}, \{S_k\}$?

If there are only finitely many, then clearly the answer is yes and $f = \sum_{j=1}^{n} S_j$. For infinitely many, the sum $\sum_{j=1}^{n} S_j$ need not converge; c.f. take $\{k\}$, $S_k(z) = \frac{1}{z+k}$. does not even converge at $z=0$.

Mittag-Leffler Thm. Let $\{a_k\}_{k=1}^{\infty}$ be a seq. of points in G, without limit pts in G, and

$$S_k(z) = \sum_{j=1}^{m_k} c_j \frac{1}{(z-a_k)^j}, \quad k=1,2,\ldots$$

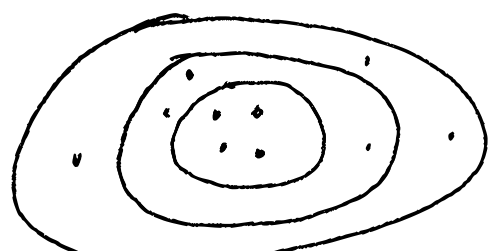
a seq. of singular parts. Then $\exists f \in \mathcal{M}(G)$ with poles and singular parts precisely given by $\{a_k\}$, $\{S_k\}$.

Pf. Let $\{K_n\}_{n=1}^{\infty}$ be an exhaustion of G by compacts w/ "standard" properties: $K_n \subset\subset \operatorname{int} K_{n+1}$, $\forall K \subset\subset G$ $\exists N$ s.t. $K \subseteq K_N$, each component of $\mathbb{C} \setminus K$ contains a component of $\mathbb{C} \setminus G$.

Let $I_1 = \{k : a_k \in K_1\}$, $I_2 = \{k : a_k \in K_n \setminus K_{n-1}\}$

...

Each I_n is finite.



Let $g_n = \sum_{k \in I_n} S_k$. Note, by construction, g_n is analytic in some open nbhd of K_{n-1}.

Let E be a set that meets every component of $\mathbb{C}_\infty \setminus G$. By Runge, we can find $R_n \in R(E)$ s.t.

$$\sup_{K_{n-1}} |g_n - R_n| < \left(\tfrac{1}{2}\right)^n \quad \text{for } n \geq 2.$$

For each N, let $f_N = g_1 + \sum_{n=2}^{N} (g_n - R_n)$.

Each f_N is meromorphic in G w/ all poles in K_N w/ prescribed singular parts S_k, $k \in \bigcup_{n=1}^{N} J_n$.

Let $K \subset\subset G$. Then, for $N \le n \le m$, $K \subset\subset \operatorname{int} K_{N-1}$

$$\sup_K |f_n - f_m| \le \sum_{k=n}^{m} \sup_{K_{N-1}} |g_k - R_k| \le \sum_{k=n}^{m} \frac{1}{2^k} \to 0 \text{ as } n, m \to \infty.$$

Since $d_\infty(\cdot,\cdot)$ (Fubini-Study) and $d = |\cdot - \cdot|$ (Euclidean) are "equivalent" on $\mathbb{C}$ (away from ∞) $\Rightarrow$ $f_n - f_m$ converge unif. on K to 0 in both $(\mathbb{C}_\infty, d_\infty)$ and $(\mathbb{C}, d)$.

Since $\mathcal{M}(G) \cup \{\infty\}$ is closed in $\mathcal{C}(G, \mathbb{C}_\infty)$ we conclude that $f_n \to f \in \mathcal{C}(G, \mathbb{C}_\infty)$ s.t. either $f \equiv \infty$ or $f \in \mathcal{M}(G)$. But if $K = \overline{B(a,\varepsilon)} \subset\subset G$ s.t. no $a_k \in K$, then the f_n are analytic in $B(a,\varepsilon)$ and then $f_n \to f \in H(B(a,\varepsilon)) \Rightarrow f \in \mathcal{M}(G)$. Moreover, since

$$f_n - \left(g_1 + \sum_{k=2}^{n} (g_k - R_k)\right) \text{ is analytic}$$

in $G_n = \operatorname{int} K_n$ and $\to$ to

$$f - \left(g_1 + \sum_{k=2}^{n} (g_k - R_k)\right) \text{ in } H(G_n)$$

it follows that the poles and

singular parts in G_n are given precisely by the $S_n(z)$ s.t. $k \in I_1 \cup \ldots \cup I_n$, as claimed. $\boxtimes$